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DIRICHLET AND NEUMANN BOUNDARY CONDITIONS FOR THE PRESSURE POISSON EQUATION OF INCOMPRESSIBLE FLOW

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SUMMARY

In a recent paper Gresho and Sani[1] showed that Dirichlet and Neumann boundary conditions for the pressure Poisson equation give the same solution. The purpose of this paper is to confirm this (for one case at least) by numerically solving the pressure equation with Dirichlet and Neumann boundary conditions for the inviscid stagnation point flow problem. The Dirichlet boundary condition is obtained by integrating the tangential component of the momentum equation along the boundary. The Neumann boundary condition is obtained by applying the normal component of the momentum equation at the boundary. In this work solutions for the Neumann problem exist only if a compatibility condition is satisfied. A consistent finite difference procedure which satisfies this condition on non-staggered grids is used for the solution of the pressure equation with Neumann conditions. Two test cases are computed. In the first case the velocity field is given from the analytical solution and the pressure is recovered from the solution of the associated Poisson equation. The computed results are identical for both Dirichlet and Neumann boundary conditions. However, the Dirichlet problem converges faster than the Neumann case. In the second test case the velocity field is computed from the momentum equations, which are solved iteratively with the pressure Poisson equation. In this case the Neumann problem converges faster than the Dirichlet problem.

KEY WORDS Incompressible Flow Pressure Poisson Equation Inviscid Flow Boundary Conditions

INTRODUCTION

The pressure Poisson equation plays two distinct roles in the formulation of the incompressible Navier–Stokes equations. First, when given the velocity field as in the streamfunction–vorticity formulation, the Poisson equation is used to calculate the pressure. Second, when the pressure equation is solved iteratively with the momentum equations it is used (1) to calculate the pressure and (2) to enforce the continuity equation.

Traditionally, the pressure Poisson equation is solved with Neumann boundary conditions obtained from applying the normal component of the momentum equation at the boundary. Solutions for the Neumann problem require the satisfaction of a compatibility condition (Green's theorem) which relates the source of the Poisson equation and the Neumann boundary conditions. Failure to satisfy this condition results in non-convergent iterative solutions[2] because no solution exists. Solutions for the Neumann problem are obtained using the consistent finite difference method of References 3 and 4 which satisfies the compatibility condition on non-staggered grids.

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The Dirichlet boundary conditions are obtained by integrating the tangential component of the momentum equation along the boundary. An excellent discussion on the pressure boundary conditions is given in Reference 1.

Numerical results are obtained for the pressure Poisson equation using both types of boundary conditions. The test case considered here is the two-dimensional inviscid stagnation point flow problem. It is chosen because of its analytical solution which facilitates verification of the numerical method. Numerical solutions for the pressure equation are obtained using the successive over-relaxation method. The computed results, when given the correct velocity field, show that the Dirichlet problem converges faster than the Neumann problem using the optimum over-relaxation parameter for each case. However, for the case when the pressure equation is solved iteratively with the momentum equation, the Neumann problem converges faster than the Dirichlet case.

GOVERNING EQUATIONS

The incompressible inviscid flow equations are written in Cartesian co-ordinates x and y as follows.

Continuity equation

$$U_x + V_y = 0. \tag{1}$$

Momentum equations

$$U_t + UU_x + VU_y = -P_x, \tag{2}$$

$$V_t + UV_x + VV_y = -P_y. \tag{3}$$

Here U, V are the velocity components in the x, y directions and P is the pressure divided by the density. The subscripts t, x, y refer to partial derivatives with respect to time and space.

Numerical solutions for equations (1)–(3) can be obtained using primitive and non-primitive variable formulations.[2] For both cases the pressure is computed from a Poisson-type equation which is derived from the divergence of the momentum equation.

Pressure Poisson equation

The pressure Poisson equation is derived by differentiating equation (2) with respect to x and equation (3) with respect to y and adding

$$P_{xx} + P_{yy} = \sigma - D_t, \tag{4}$$

where

$$-\sigma = (UU_x + VU_y)_x + (UV_x + VV_y)_y \tag{4a}$$

and

$$D = U_x + V_y. \tag{4b}$$

The pressure equation (4) plays two roles in the solution of equations (1)–(3) using primitive and non-primitive variable formulations.

Non-primitive variable formulation

In the non-primitive solutions of equations (1)–(3) the pressure is eliminated from the momentum equations (2) and (3). This requires the use of new dependent variables such as streamfunction and vorticity. After the velocity field is computed, the pressure is then recovered from the solution of equation (4). The unsteady term D_t in this case is eliminated from the right-hand side of equation (4) because the continuity equation is already satisfied through the use of a streamfunction as in the streamfunction–vorticity formulation or by other means as in the velocity–vorticity formulation.[2]

Primitive variable formulation

In this case the momentum equations (2) and (3) are solved for the velocity components U and V by marching in time. At each time step the pressure is computed from equation (4), which serves two functions: (i) it computes the pressure and (ii) it enforces the continuity equation as follows. The unsteady term D_t on the right-hand side of equation (4) is approximated by

$$D_t = \frac{D^{n+1} - D^n}{\Delta t}, \tag{5}$$

where D^{n+1} is set equal to zero to satisfy the continuity equation (1) at the time level $t + \Delta t$.[2]

BOUNDARY CONDITIONS

Boundary conditions are discussed with reference to the two-dimensional stagnation point flow problem shown in Figure 1. Dirichlet boundary conditions for the velocity field are specified along all the boundaries and are obtained from the analytical solution $U = x$ and $V = -y$. Two types of boundary conditions for the pressure are obtained by applying the momentum equations (2) and (3) along the boundary contour.

Neumann boundary conditions

The Neumann conditions are obtained from the normal component of the momentum equation. With reference to Figure 1 the following conditions are used:

$$P_x = -UU_x - VU_y \qquad \text{at } x = 1, \tag{6a}$$

$$P_y = -UV_x - VV_y \qquad \text{at } y = 0,\ 1. \tag{6b}$$

The symmetry condition is used at $x = 0$.

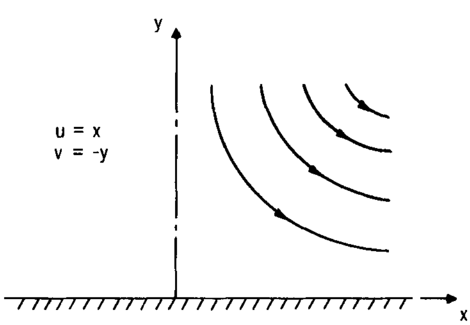

Figure 1. The inviscid stagnation flow problem

Solutions for equation (4) with the boundary conditions equation (6) require the satisfaction of a compatibility condition resulting from Green's theorem.

Compatibility condition

$$\iint_A \sigma \, \mathrm{d}A = \oint_s P_n \, \mathrm{d}s, \tag{7}$$

where s is the boundary contour enclosing the area of the solution domain A, $P_n = \mathbf{n} \cdot \nabla P$, and $\mathbf{n}$ is the outward normal to the boundary s.

Failure to satisfy the compatibility condition (7) results in non-convergent iterative solutions for equation (4)[2] because no solution exists. The consistent finite difference method of References 3 and 4, which satisfies the compatibility condition on non-staggered grids, is used to calculate the pressure.

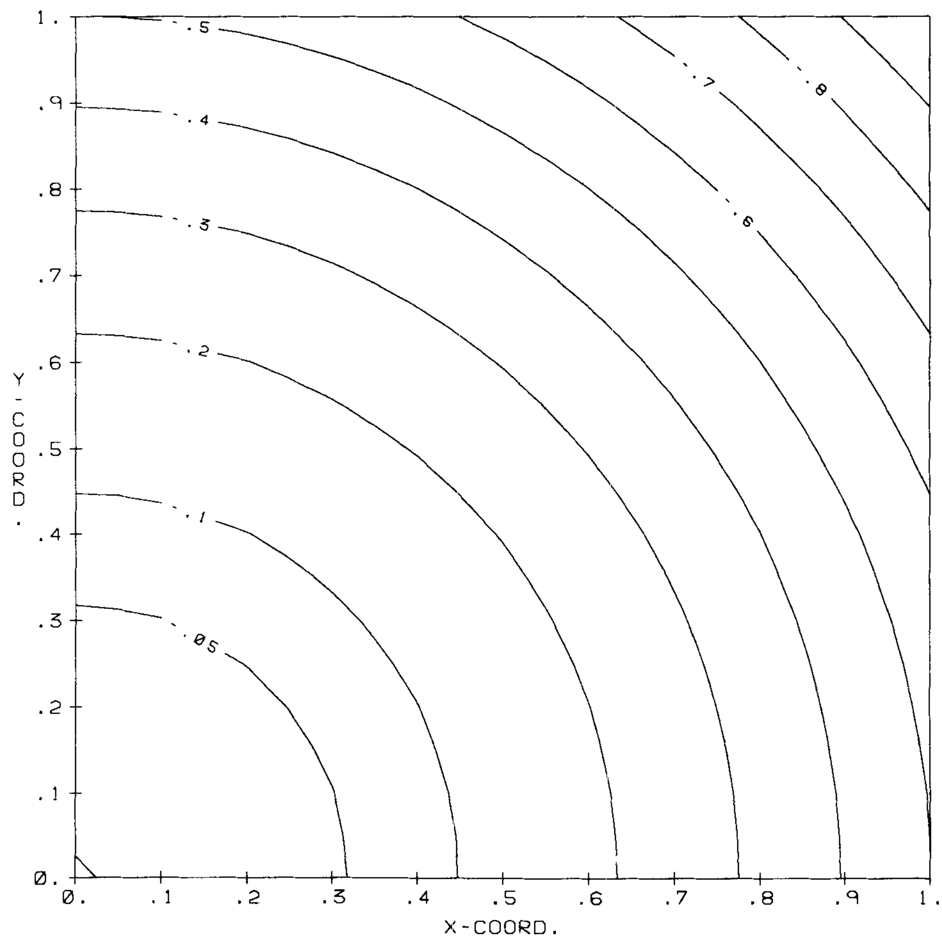

Figure 2. Computed static pressure contours

Dirichlet boundary conditions

The Dirichlet conditions for the pressure are obtained by integrating the tangential component of the momentum equation along the boundary for potential flow:

$$P = -\tfrac{1}{2}(U^2 + V^2) \quad \text{at } y=0,\ y=1 \text{ and } x=1. \tag{8}$$

The symmetry condition is used at $x=0$.

NUMERICAL SOLUTIONS

Numerical solutions for the pressure Poisson equation (4) with Neumann and Dirichlet boundary conditions (6) and (8) are obtained. The Neumann problem is solved using the method of References 3 and 4. No special treatment is needed for the Dirichlet problem. The classical second-order central finite difference approximations are used to approximate the Laplace

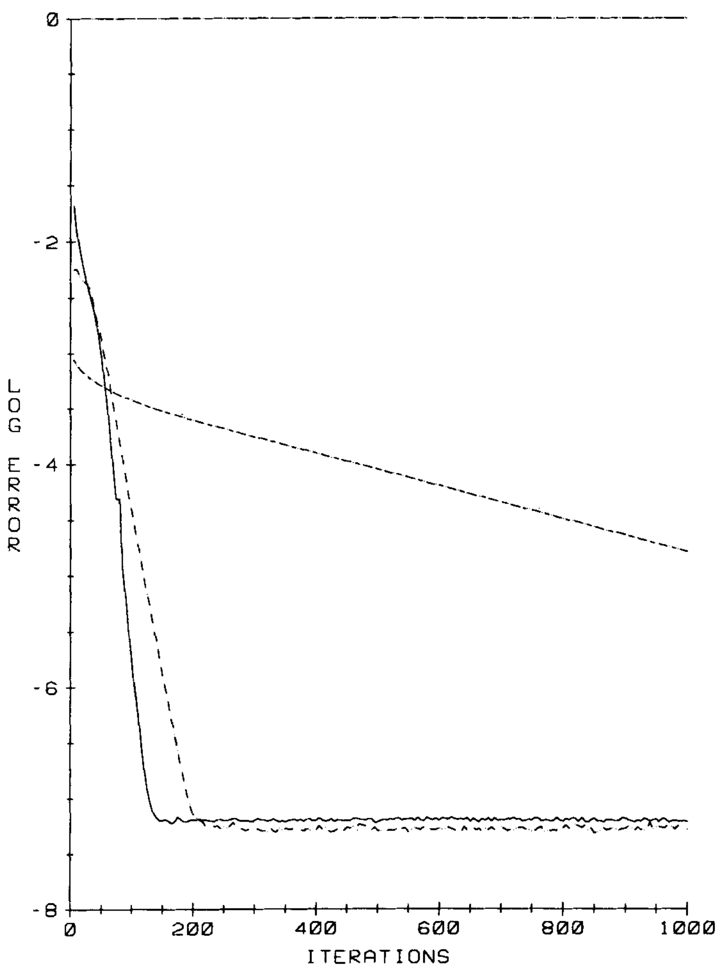

Figure 3. Convergence characteristics for (– – –) Neumann, (– · –) incorporated Neumann and (——) Dirichlet boundary conditions

operator and the source term in equation (4). The method of successive over-relaxation is employed in the numerical solutions of equation (4).

Both roles of the pressure Poisson equation in the formulation of equations (1)–(3) are considered. First, given the velocity field from the analytical solution, the pressure equation (4) is solved for the pressure with Neumann and Dirichlet boundary conditions. The computed pressure field, which is identical for both Dirichlet and Neumann boundary conditions, is shown in Figure 2. The computed results are obtained using 40 × 40 non-staggered grids in the x, y directions. The maximum error is less than 2%.

The convergence characteristics for the Neumann and Dirichlet problems are shown in Figure 3 for over-relaxation parameters 1·95 and 1·85 respectively. It can be seen from Figure 3 that the Dirichlet problem converges faster than the Neumann Problem. Direct incorporation of the Neumann boundary conditions in the differential equation at grid points next to the boundary is also considered.[2] It is found that over-relaxation parameters greater than one cause divergence, while those less than one cause slow convergence. This option is not recommended here because of its poor stability and convergence behaviour as shown in Figure 3.

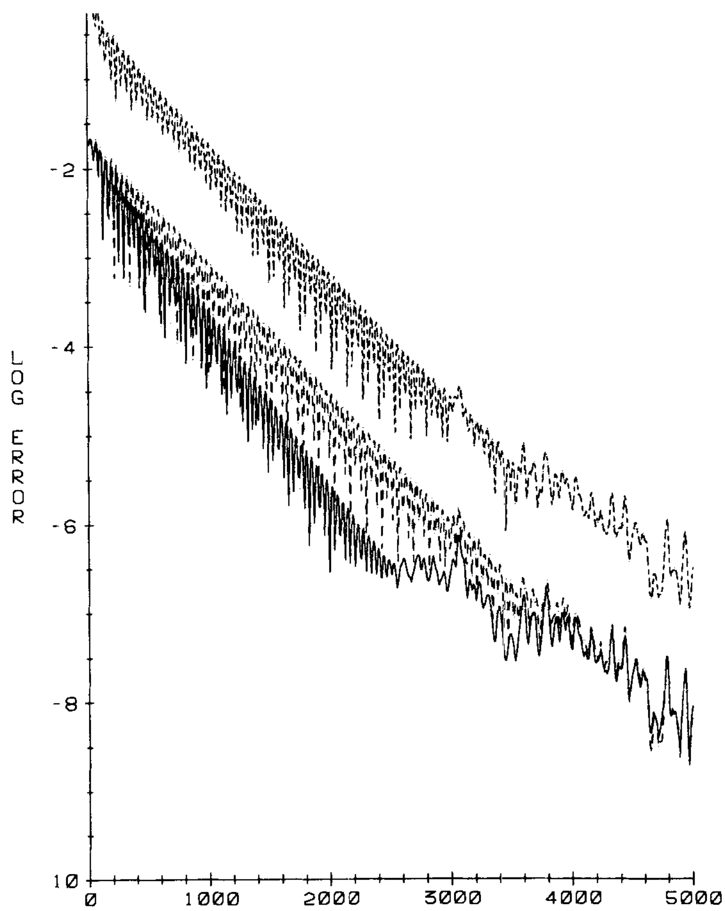

Figure 4. Convergence characteristics for Neumann boundary conditions in the primitive variable formulation: (—) u-velocity, (– – –) v-velocity and (– ·–) pressure

Second, the velocity field is computed from the momentum equations (2) and (3) by marching in time using the explicit first-order upwind differencing scheme. A reasonably fine grid (40 x 40 grid points) is used here to reduce the artificial viscosity effect introduced by the upwind scheme. The pressure equation (4) with Neumann and Dirichlet boundary conditions is solved iteratively at each time step. The successive over-relaxation method is used for the numerical solution of equation (4), using over-relaxation parameters 1·35 and 1·25 respectively for the Neumann and Dirichlet problems. Only one iteration is employed for the solution of the pressure equation at each time step. We should mention here that initial conditions for both pressure and velocity are set equal to zero except at the boundary. Thus the transient problem is not well posed.[1] However, when the solution converges to the steady state the problem becomes well posed.[5]

The computed pressure contours using 40 x 40 grid points are identical with those of Figure 2. The convergence characteristics for the velocity and pressure are shown in Figure 4 for the Neumann problem. The iteration number in Figure 4 refers to the number of time steps in the admittedly meaningless transient. Similar results are shown in Figure 5 for the Dirichlet case. It

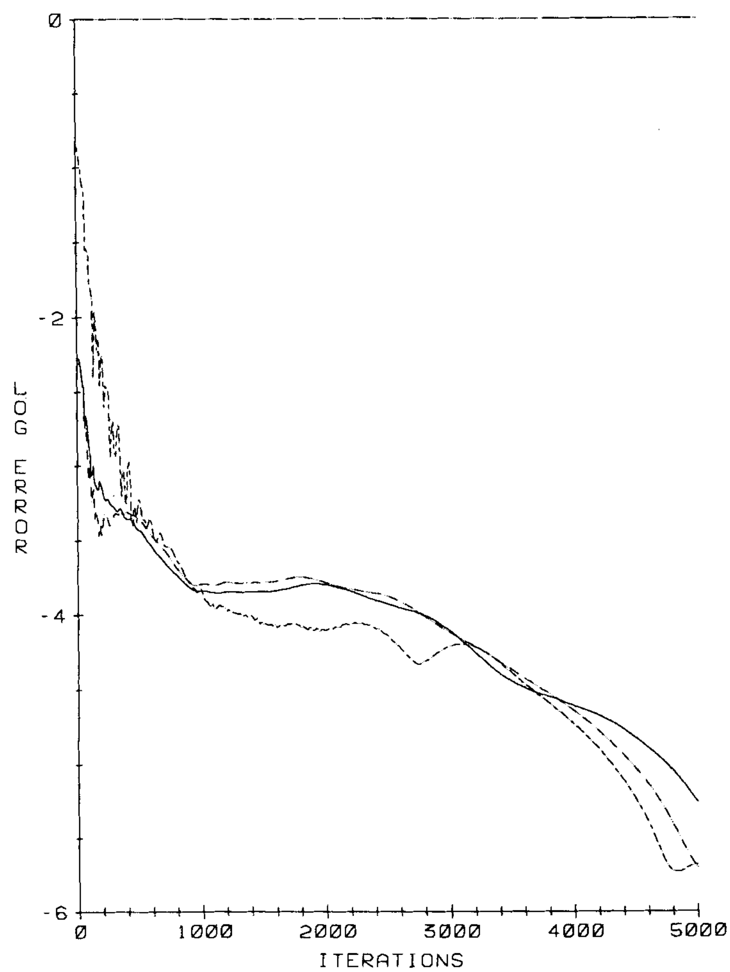

Figure 5. Convergence characteristics for Dirichlet boundary conditions in the primitive variable formulation: (——) u-velocity, (– – –) v-velocity and (– · –) pressure

can be seen from Figures 4 and 5 that the Neumann problem converges faster than the Dirichlet problem for the primitive variable formulation.

CONCLUSIONS

Numerical solutions for the pressure Poisson equation are identical for both Dirichlet and Neumann boundary conditions. This confirms the analytical discussion of Reference 1 on pressure boundary conditions, at least for the case studied here. The Dirichlet boundary condition requires an additional integration step of the tangential momentum equation along the boundary. This inconvenient process might be further complicated by the presence of singular points on the boundary. Also, depending on the method of integration, discontinuities in the pressure boundary conditions might arise.

In the non-primitive variable formulations the use of Dirichlet boundary conditions for the pressure accelerates the convergence rate. However, the Neumann problem converges faster for the primitive variable formulation.

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